II Mathematical Tools (Dynamic Optimization

(I) Introduction to Dynamic Optimization

1 Examples of Dynamic Optimization Problems

- A Discrete-Variable Example
- A Continuous-Variable Example

2 Forms of Dynamic Optimization Problems

2.1 Forms of Objective Functional

- Standard form: $V[y] = \int_0^T F[t, y(t), y'(t)]dt$
- Terminal-Control Problem (Problem of Mayer): $V[y] = G[T, y(T)]$
- Problem of Bolza: $V[y] = \int_0^T F[t, y(t), y'(t)]dt + G[T, y(T)]$

2.2 Types of Terminal Points

- Fixed-Terminal-Point Problem: both T and Z are fixed
- Fixed-Time (Vertical-Terminal-Line) Problem: T is fixed, but Z is free
- Fixed-Endpoint (Horizontal-Terminal-Line) Problem: T is free, but Z is fixed
- Terminal-Curve Problem: both T and Z are free

3 Alternative Approaches

- Calculus of Variations
- Dynamic Programming
- Optimal Control Theory

(II) Optimal Control Theory

1 The Maximum Principle

1.1 The Simplest Problem

$$\max \quad V[y] = \int_0^T F[t, y, u]dt$$

$$\text{s.t.} \quad \dot{y} = f(t, y, u) \quad \text{(Equation of motion)}$$

$$y(0) = A, \quad y(T) \ \text{ free} \quad (A, T \text{ given})$$

$$\text{and} \quad u(t) \in \mathcal{U} \quad \forall t \in [0, T]$$

where y is a *state* variable and u is a *control* variable. Note that the simplest problem is a vertical-terminal-line problem.

1.2 The Maximum Principle (L. S. Pontryagin)

The *Hamiltonian function*

$$H(t, y, u, \lambda) \equiv F(t, y, u) + \lambda(t) f(t, y, u)$$

where λ is a *costate* variable. The maximum principle conditions

$$\max_u H(t, y, u, \lambda) \quad \forall t \in [0, T]$$

$$\dot{y} = \frac{\partial H}{\partial \lambda} \quad \text{(Equation of motion for } y\text{)}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial y} \quad \text{(Equation of motion for } \lambda)$$

$$\lambda(T) = 0 \quad \text{(TVC)}$$

1.3 Alternative Terminal Conditions

- Fixed terminal Point

$$y(T) = y_T \quad (T, y_T \text{ given}) \quad \text{(TVC)}$$

- Horizontal terminal Line (Fixed-Endpoint Problem)

$$[H]_{t=T} = 0$$

- Terminal Curve

$$[H - \lambda\phi']_{t=T} = 0 \quad \text{(TVC)}$$

- Truncated Vertical Terminal Line

$$\lambda(T) \geq 0 \;\; y_T^* \leq y_{min} \;\; (y_T - y_{min})\lambda(T) = 0 \quad \text{(TVCs)}$$

- Truncated Horizontal Terminal Line

$$[H]_{t=T} \geq 0 \;\; T^* \leq T_{max} \;\; (T^* - T_{max})\,[H]_{t=T} = 0 \;\; \text{(TVCs)}$$

1.4 The Current-Value Hamiltonian

If F function can be written as $F(t, y, u) = G(t, y, u)e^{-\rho t}$, then we define the *current Hamiltonian function* as

$$H_c \equiv He^{\rho t} = G(t, y, u) + m(t)f(t, y, u)$$

The revised maximum principle conditions

$$\max_u H_c \quad \forall t \in [0, T]$$

$$\dot{y} = \frac{\partial H_c}{\partial m} \quad \text{(Equation of motion for } y\text{)}$$

$$\dot{m} = -\frac{\partial H_c}{\partial y} + \rho m \quad \text{(Equation of motion for } m\text{)}$$

$$m(T)e^{-\rho T} = 0 \quad \text{(TVC)}$$

The revised TVCs

- Fixed terminal Point

$$y(T) = y_T \quad (T, y_T \text{ given}) \quad \text{(TVC)}$$

- Horizontal terminal Line (Fixed-Endpoint Problem)

$$[H_c]_{t=T}\, e^{-\rho T} = 0$$

- Terminal Curve

$$[H_c - m\phi']_{t=T}\, e^{-\rho T} = 0 \quad \text{(TVC)}$$

- Truncated Vertical Terminal Line

$$m(T)e^{-\rho T} \geq 0 \;\; y_T^* > y_{min} \;\; (y_T - y_{min})m(T)e^{-\rho T} = 0 \quad \text{(TVCs)}$$

- Truncated Horizontal Terminal Line

$$[H_c]_{t=T} \geq 0 \;\; T^* \leq T_{max} \;\; (T^* - T_{max})\,[H_c]_{t=T} = 0 \;\; \text{(TVCs)}$$

An example:

$$\max \int_0^T e^{-\rho t}[c(t)]^\gamma dt, \quad 0 < \gamma < 1$$

$$\text{s.t.} \quad \dot{k}(t) = A[k(t)]^\alpha - c(t), \quad 0 < \alpha < 1$$

where $k(0)$ and T are given and $K(T)$ is free.

1.5 Problems with n State Variables and m Control Variables

The optimal control problem

$$\text{Max} \;\; V = \int_0^T F[t, y_1, ..., y_n, u_1, ..., u_m]dt$$

$$\text{s.t.} \;\; \dot{y}_j = f^j(t, y_1, ..., y_n, u_1, ..., u_m) \quad \text{(Equation of motion)}$$

$$y_j(0) = y_{j0}, \;\; y_j(T) = y_{jT}$$

$$\text{and} \;\; u_i(t) \in \mathcal{U}_i \;\; (i = 1, ..., m, j = 1, ..., n)$$

The Hamiltonian function

$$H \equiv F(t, \mathbf{y}, \mathbf{u}) + \lambda' \mathbf{f}(t, \mathbf{y}, \mathbf{u})$$

The maximum principle conditions and TVCs

$$\max_{\mathbf{u}} H$$

$$\frac{\partial H}{\partial y_j} = -\dot{\lambda}_j \quad (j = 1, ...n)$$

$$\frac{\partial H}{\partial \lambda_j} = -\dot{y}_j \quad (j = 1, ...n)$$

$$[H]_{t=T}\Delta T - \sum_{j=1}^{n} \lambda_j(T)\Delta y_{jT} = 0 \quad \text{(General TVC)}$$

2 Infinite-Horizon Problem

The optimal control problem

$$\text{Max } V = \int_0^\infty F(t, y, u)dt$$

$$\text{s.t. } \dot{y} = f(t, y, u)$$

$$y(0) = y_0, \ \ y_0 \text{ given}$$

2.1 The TVCs (Michel 1982)

$$\lim_{t\to\infty} H = 0 \ \ \text{(Infinite-horizon TVC)}$$

$$\lim_{t\to\infty} \lambda(t) = 0 \ \ \text{(TVC for free terminal state)}$$

2.2 Applications

(A). Optimal Growth (Cass, *RES*, 1965)

(B). Endogenous Technological Change (Romer, *JPE*, 1990)

(C). On the Mechanics of Economic Development (Lucas, *JME*, 1988)

3 Optimal Control with Constraints

3.1 Types of Constraints

A. Constraints Involving Control Variables

- Equality Constraints
- Inequality Constraints
- Equality Integral Constraints
- Inequality Integral Constraints

B. State-Space Constraints

3.2 First-Order Conditions

- Equality Constraints

$$\max V = \int_0^T F(t, y, u_1, u_2)dt$$

$$\text{s.t. } \dot{y} = f(t, y, u_1, u_2)$$

$$g(t, y, u_1, u_2) = c, \quad c \text{ constant}$$

and boundary conditions. Note that the number of constraints constraints should be smaller than the number of control variables. We construct a Lagrangian function

$$\mathcal{L} = H + \theta(t)(c - g)$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial u_j} = 0 \ \forall t \in [0, T] \ \ (j = 1, 2)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 \ \ \forall t \in [0, T]$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \dot{y}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -\dot{\lambda}$$

- Inequality Constraints

$$\max V = \int_0^T F(t, y, u_1, u_2) dt$$

$$\text{s.t. } \ \dot{y} = f(t, y, u_1, u_2)$$

$$g^i(t, y, u_1, u_2) \leq c_i, \ \ c_i \ \text{ constant}$$

and boundary conditions. Similar to the equality constraint case, we construct the Lagrangian function

$$\mathcal{L} = F + \lambda f + \sum_{i=1}^{2} \theta_i (c_i - g^i)$$

Then the first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial u_j} = 0 \ \ \forall t \in [0, T] \ \ (j = 1, 2)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_i} \geq 0 \quad \theta_i \geq 0 \text{ and } \theta_i \frac{\partial \mathcal{L}}{\partial \theta_i} = 0$$

$$(i = 1, 2 \text{ and } j = 1, 2) \ \forall t \in [0, T]$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \dot{y}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -\dot{\lambda}$$

- Equality Integral Constraints (Isoperimetric Problems)

$$\max V = \int_0^T F(t, y, u) dt$$

$$\text{s.t. } \dot{y} = f(t, y, u)$$

$$\int_0^T G(t, y, u) dt = k \ \ k \text{ constant}$$

and boundary conditions. Define a new state variable

$$\Gamma(t) \equiv -\int_0^t G(t, y, u) dt,$$

then

$$\dot{\Gamma} = -G(t, y, u)$$

$$\text{with } \Gamma(0) = 0 \text{ and } \Gamma(T) = -k$$

Now the optimization problem becomes

$$\max V = \int_0^T F(t, y, u)dt$$

$$\text{s.t. } \dot{y} = f(t, y, u)$$

$$\dot{\Gamma} = -G(t, y, u)$$

and boundary conditions. We construct the following Hamiltonian function

$$H = F(t, y, u) + \lambda f(t, y, u) - \mu G(t, y, u)$$

- Inequality Integral Constraints

$$\max V = \int_0^T F(t, y, u)dt$$

$$\text{s.t. } \dot{y} = f(t, y, u)$$

$$\int_0^T G(t, y, u)dt \leq k \quad k \text{ constant}$$

and boundary conditions. The solution approach is the same as in the Equality Integral Constraint case. But now the we have a truncated vertical terminal line problem, where $\Gamma(T) \geq -k$.

(III) Dynamic Programming

1 An Example and the Principle of Optimality

1.1 An Example: The Shortest Distance Problem

1.2 The Principle of Optimality

An optimal path has the property that whatever the inital conditions and control values over some inital period, the control (or decision) variables over the remaining period must be optimal for the remaining problem, with the state resulting from the early decisions considered as the inital conditions.

2 Continuous Time Problems

2.1 The Optimization Problem and the Optimality Condition

$$\max \int_0^T F(t, y, u) dt$$

$$\text{s.t. } \dot{y} = f(t, y, u), \quad y(0) = y_0$$

Define the optimal value function $J(t, y)$ as the best value that can be obtained starting at time t in state y. Then the value function $J(t, y)$ obeys the following *Bellman equation*

$$-\frac{\partial J(t, y)}{\partial t} = \max_u [F(t, y, u) + \frac{\partial J(t, y)}{\partial y} f(t, y, u)]$$

2.2 Derivation of the Optimality Condition

From the definition of $J(t, y)$, we have:

$$J(t_0, y_0) = \max_u \int_{t_0}^T F(t, y, u) dt$$

$$\text{s.t. } \dot{y} = f(t, y, u), \quad y(0) = y_0$$

The value function can be rewritten as

$$J(t_0, y_0) = \max_u \left[\int_{t_0}^{t_0+\Delta t} F dt + \int_{t_0+\Delta t}^{T} F \right]$$

where Δt is taken to be very small and positive. The control function $u(t)$, $t_0 + \Delta t \le t \le T$, should be optimal for the problem begining at $t_0 + \Delta t$ in state $y(t_0 + \Delta t) = y_0 + \Delta y$. The state $y(t_0 + \Delta t)$ depends on the state y_0 and the control function $u(t)$ chosen over the period $t_0 \le t \le t_0 + \Delta t$. Now we rewrite $J(t_0, y_0)$ as

$$J(t_0, y_0) = \underbrace{\max_u}_{t_0 \le t \le t_0+\Delta t} \left[\int_{t_0}^{t_0+\Delta t} F dt + \underbrace{\max_u}_{t_0+\Delta t \le t \le T} \int_{t_0+\Delta t}^{T} F \right]$$

$$\text{s.t. } \dot{y} = f(t, y, u), \quad y(t_0 + \Delta t) = y_0 + \Delta y$$

Or equivalently

$$J(t_0, y_0) = \underbrace{\max_u}_{t_0 \le t \le t_0+\Delta t} \left[\int_{t_0}^{t_0+\Delta t} F dt + J(t_0 + \Delta t, y_0 + \Delta y) \right]$$

Assume that $J(t, y)$ is twice continuously differentiable. Using Taylor expansion, we have

$$J(t_0, y_0) = \max_u \left[F(t_0, y_0, u)\Delta t + J(t_0, y_0) + \frac{\partial J(t_0, y_0)}{\partial t} \Delta t \right.$$

$$\left. + \frac{\partial J(t_0, y_0)}{\partial y} \Delta y + h.o.t. \right]$$

Dividing through by Δt and letting $\Delta t \to 0$ gives the above optimality condition.

2.3 Infinite-Horizon Autonomous Problems

The infinite horizon autonomous problem:

$$\max \int_0^\infty e^{-rt} G(y,u)dt$$
$$\text{s.t. } \dot{y} = f(y,u), \;\; y(0) = y_0$$

Then

$$J(t_0, y_0) = \max_u \int_{t_0}^\infty e^{-rt_0} G(y,u)dt$$
$$= e^{-rt_0} \max_u \int_{t_0}^\infty e^{r(t-t_0)} G(y,u)dt$$

Note that the value of the integral on the RHS depends on the initial state, but is independent of the inital time. Let the current value function be

$$V(y_0) = \max_u \int_{t_0}^\infty e^{r(t-t_0)} G(y,u)dt$$

Then

$$J(t,y) = e^{-rt}V(y), \qquad \frac{\partial J(t,y)}{\partial t} = -re^{-rt}V(y)$$
$$\frac{\partial J(t,y)}{\partial y} = e^{-rt}V'(y)$$

Then we have a simpler form of the optimality condition (Bellman equation)

$$rV(y) = \max_u [G(y,u) + V'(y)f(y,u)]$$

2.4 An Example

$$\max \int_0^\infty e^{-\rho t} \ln c(t), \quad \rho > 0$$
$$\text{s.t. } \dot{k}(t) = rk(t) - c(t), \;\; k(0) = k_0 \text{ given}$$

3 Discrete Time Problems

3.1 The Optimization Problem and Bellman Equation

The maximization problem

$$\max_{u_t} \sum_{t=0}^{T} F_t(y_t, u_t)$$

$$\text{s.t. } y_{t+1} = f_t(y_t, u_t), \quad y_0 \text{ given}, \quad t = 0, 1, 2, ..., T$$

Define the value function associated with the maximization problem starting from time t with initial state y_t as

$$J(t, y_t) = \max_{u_s} \sum_{s=t}^{T} F_s(y_s, u_s)$$

$$\text{s.t. } y_{t+1} = f_t(y_t, u_t), \quad t = 0, 1, 2, ..., T$$

The value function can be rewritten as a Bellman equation

$$J_{j+1}(y_{T-j}) = \max_{u_{T-j}} \left[F_{T-j}(y_{T-j}, u_{T-j}) + J_j(y_{T-j+1})\right]$$

subject to $y_{T-j+1} = f_{T-j}(y_{T-j}, u_{T-j})$, where y_{T-j} given and $j = 0, 1, ..., T$

The Bellman equation allows us to work backward and solve the maximization problem recursively. The solutions are $u_{T-j} = h_{T-j}(y_{T-j})$ and $y_{T-j+1} = g_{T-j}(y_{T-j})$.

3.2 Discounted Dynamic Programming Problems

The maximization problem

$$\max_{u_t} \sum_{t=0}^{T} \beta^t G(y_t, u_t)$$

$$\text{s.t. } y_{t+1} = f(y_t, u_t), \quad y_0 \text{ given}$$

The corresponding Bellman equation is

$$J_{j+1}(y_{T-j}) = \max_{u_{T-j}} \left[\beta^{T-j} G(y_{T-j}, u_{T-j}) + J_j(y_{T-j+1})\right]$$

Now define the current value function

$$V_{j+1}(y_{T-j}) = \beta^{j-T} J_{j+1}(y_{T-j})$$

Then we rewrite the Bellman equation as

$$V_{j+1}(y_{T-j}) = \max_{u_{T-j}} [G(y_{T-j}, u_{T-j}) + \beta V_j(y_{T-j+1})]$$

subject to $y_{T-j+1} = f(y_{T-j}, u_{T-j})$ and y_{T-j} given. More compactly, the above equation can be rewritten as

$$V_{j+1}(y) = \max_u [G(y, u) + \beta V_j(\tilde{y})]$$

subject to $\tilde{y} = f(y, u)$ and y given. Under particular conditions (G is concave and bounded and the set $\{y_{t+1}, y_t, u_t : y_{t+1} \leq f(y_t, u_t)\}$ for admissible u_t is convex and compact), starting from any bounded and continuous initial V_0, V_j converges to V as $j \to \infty$, where $V = \lim_{j\to\infty} V_j$. That is,

$$V(y) = \max_u [G(y, u) + \beta V(\tilde{y})]$$

where $\tilde{y} = f(y, u)$. The limiting value function V is the optimal value function for the following infinite horizon problem

$$\max \sum_{t=0}^{\infty} \beta^t G(y_t, u_t)$$

$$\text{s.t. } y_{t+1} = f(y_t, u_t), \quad y_0 \text{ given}$$

There is a unique and time-invariant optimal policy of the form $u_t = h(y_t)$. The value function V is differentiable with

$$V'(y) = \frac{\partial G}{\partial y}[y, h(y)] + \beta \frac{\partial f}{\partial y}[y, h(y)] V'(f[y, h(y)])$$

This is the formula of Benveniste and Scheinkman (*Econometrica*, 1979).

3.3 Discounted Stochastic Dynamic Programming Problems

The maximization problem

$$\max \quad E_0 \sum_{t=0}^{\infty} \beta^t G(y_t, u_t)$$

$$\text{s.t. } y_{t+1} = f(y_t, u_t, \epsilon_t), \quad y_0 \text{ given}$$

where $E_t(x)$ denotes the mathematical expectation of a random variable x (given information known at time t) and $_t$ is a sequence of independently and identically distributed random variables. The Bellman equation corresponding to this problem is

$$V(y) = \max_u \{G(y, u) + \beta E[V(\tilde{y})|y]\}.$$

3 Application: Saving and Optimal Growth

3.1 Saving under Certainty

Consider the problem of a consumer who seeks to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

$$\text{s.t. } A_{t+1} = R_t(A_t + y_t - c_t), \ \ A_0 \text{ given}$$

where A_t is nonlabor wealth at the beginning of time t, y_t is exogenously given labor income at time t, R_t is one-period gross rate of return on nonlabor wealth and c_t is consumption at time t. Assume that $y_t = \lambda y_{t-1}$ (λ is the growth rate of income), $R_t = R > 0$ for all t and $R > \lambda > 0$. To rule out a strategy of infinite consumption supported by unbounded borrowing, the following isoperimetric condition is imposed:

$$\sum_{j=0}^{\infty} R^{-j} c_{t+j} = A_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j} = A_t + \sum_{j=0}^{\infty} (\lambda R^{-1})^j y_t$$

Define the state variables as (A_t, y_t, R_{t-1}) and the control variable as $u_t = R_t^{-1}A_{t+1} = A_t + y_t - c_t$. Then $A_{t+1} = R_t u_t$. Bellman equation for this problem is

$$V(A_t, y_t, R_{t-1}) = \max_{u_t}\{u(A_t + y_t - u_t) + \beta V(u_t R_t, y_{t+1}, R_t\}$$

Then the first-order condition is

$$-u'(c_t) + \beta R_t u'(c_{t+1}) = 0$$

Suppose that $u(c_t) = \ln c_t$, then we have $c_{t+j} = (\beta R)^j c_t$. Then the above isoperimetric condition implies

$$c_t = (1-\beta)\left[A_t + \frac{y_t}{1-\lambda R^{-1}}\right]$$

where $\lambda R^{-1} < 1$.

3.2 Optimal Growth

An agent aims to maximize

$$\sum_{t_0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

$$\text{s.t. } c_t + k_{t+1} = f(k_t), \quad k_0 > 0 \text{ given}$$

where $u(c_t) = \ln c_t$ and $f(k_t) = Ak_t^\alpha$ $(A > 0, 0 < \alpha < 1)$. Define the state variable as k_t and the control variable as k_{t+1}, then Bellman equation for this problem is

$$V(k_t) = \max_{k_{t+1}}\{u(Ak_t^\alpha - k_{t+1}) + \beta V(k_{t+1})\}$$

Then the first-order condition is

$$-\frac{1}{Ak_t^\alpha - k_{t+1}} + \beta V'(k_{t+1}) = 0$$

Now we use guess-and-verify method to solve this problem. [There are two classes of specifications of preferences and constraints for which this method can be used: (a) quadratic preferences and linear constraints and (b) logarithmic preferences and Cobb-Douglass constraints.] We make the guess

$$V(k) = E + F \ln k$$

where E and F are undetermined coefficients. Then the first-order condition implies

$$\tilde{k} = \left(\frac{\beta F}{1 + \beta F}\right) A k^{\alpha}$$

Substituting this into Bellman equation gives

$$F = \frac{\alpha}{1 - \alpha\beta}$$

$$E = (1 - \beta)^{-1} \left[\ln A(1 - \alpha\beta) + \left(\frac{\alpha\beta}{1 - \alpha\beta}\right) \ln A\alpha\beta\right]$$

Note that k_t converges to $k_\infty = (A\alpha\beta)^{1/(1-\alpha)}$ as $t \to \infty$ for any initial value k_0.

3.3 Stochastic Optimal Growth

A planner seeks to maximize

$$E_0 \sum_{t_0}^{\infty} \beta^t \ln(c_t), \quad 0 < \beta < 1$$

$$\text{s.t. } c_t + k_{t+1} = A k_t^{\alpha} \theta_t, \;\; k_0 > 0 \text{ given}$$

where $A > 0$ and $0 < \alpha < 1$. Assume that $\ln \theta_t$ is an independently and identically distributed random variable with normal distribution, $N(0, \sigma^2)$. Define the state variables as (k_t, θ_t) and the control variable as k_{t+1}, then Bellman equation for this problem is

$$V(k_t, \theta_t) = \max_{k_{t+1}} \{u(A k_t^{\alpha} \theta_t - k_{t+1}) + \beta E[V(k_{t+1}, \theta_{t+1}) | k_t, \theta_t\}$$

Then the first-order condition is

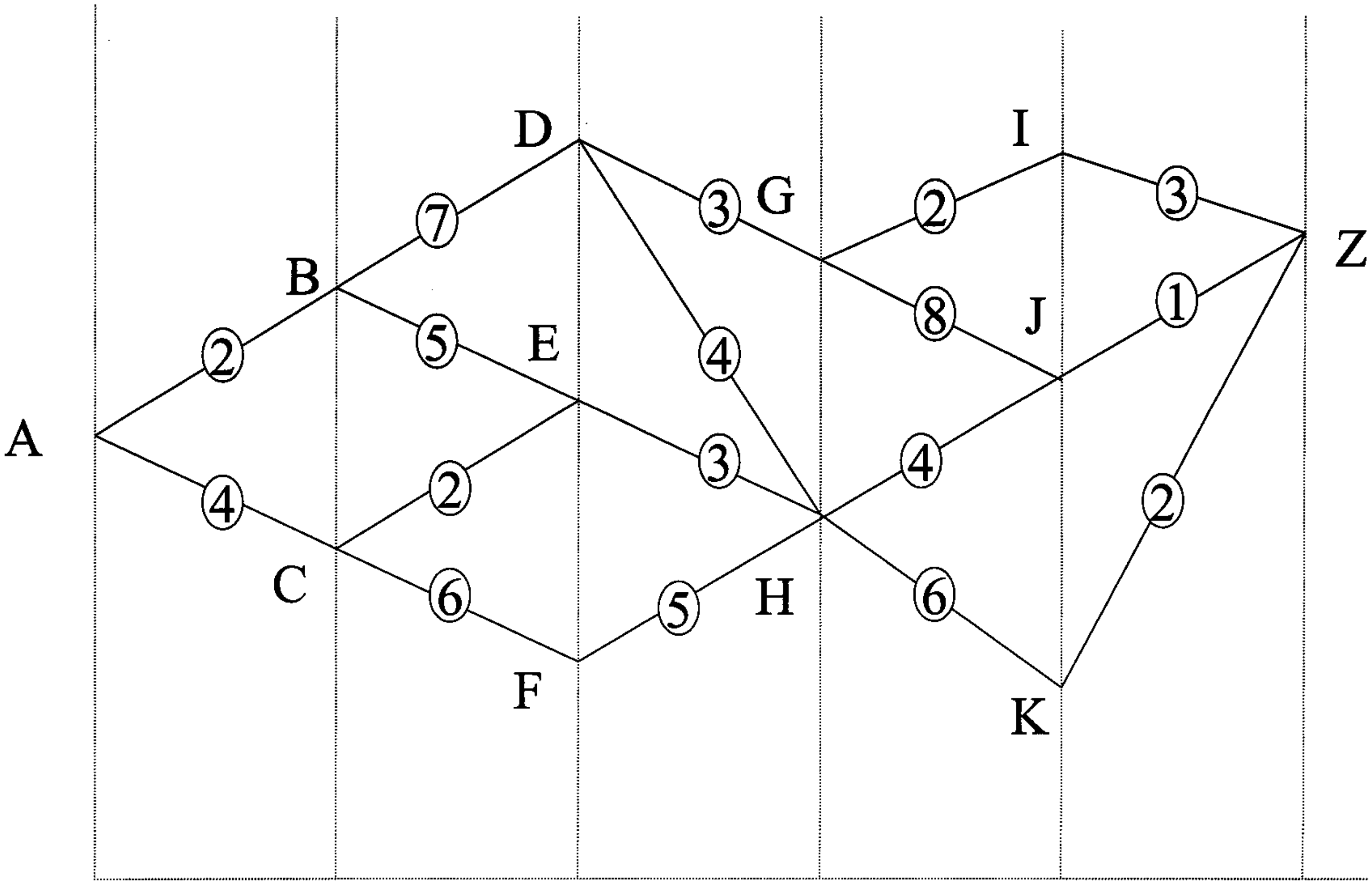
$$-\frac{1}{Ak_t^\alpha - k_{t+1}} + \beta V'(k_{t+1}) = 0$$

Guess that the value function is

$$V(k, \theta) = E + F \ln k + G \ln \theta$$

where E, F and G are undetermined coefficients. Then the optimal policy rule is

$$k_{t+1} = A\alpha\beta k_t^\alpha \theta_t.$$



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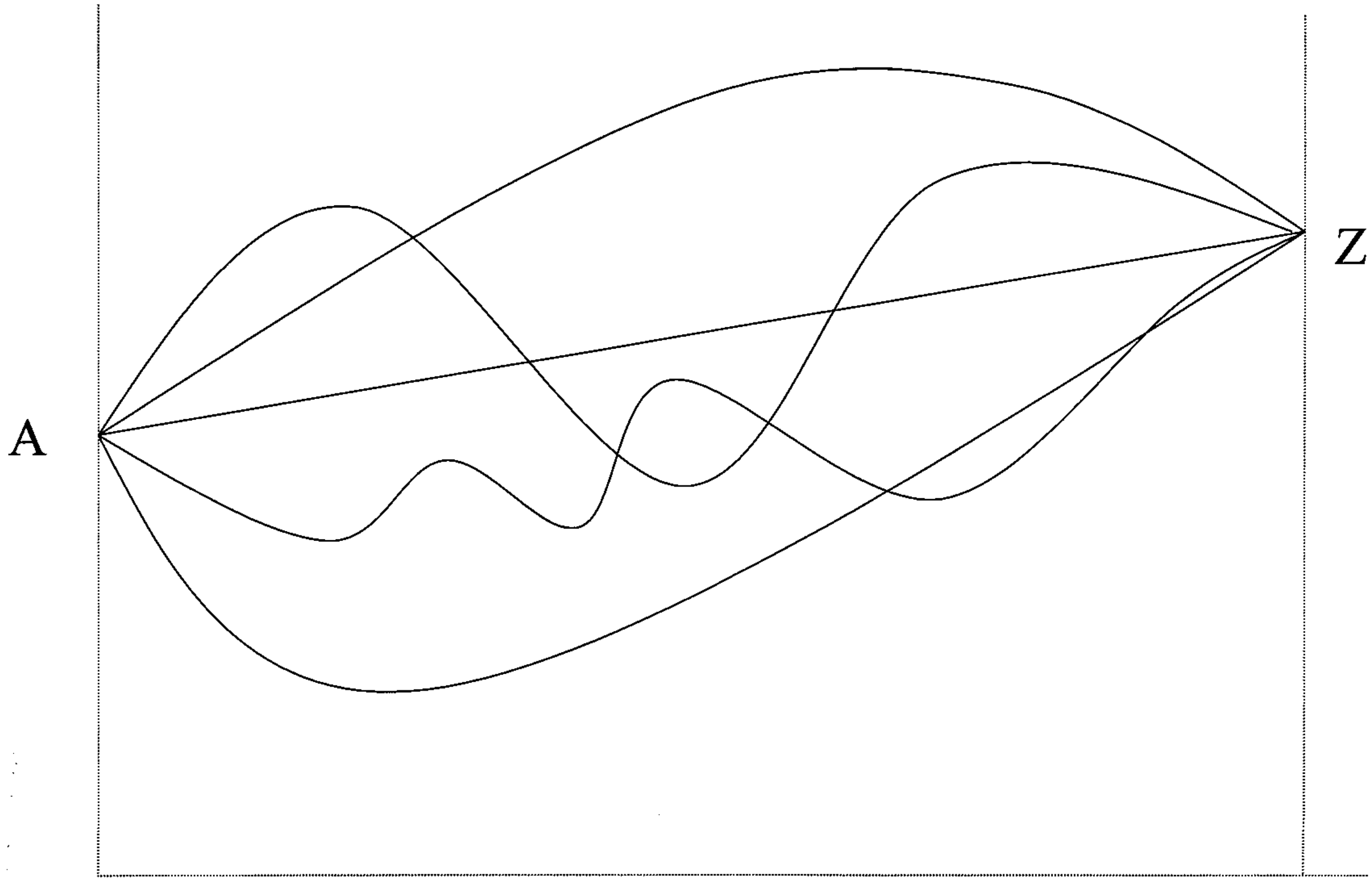
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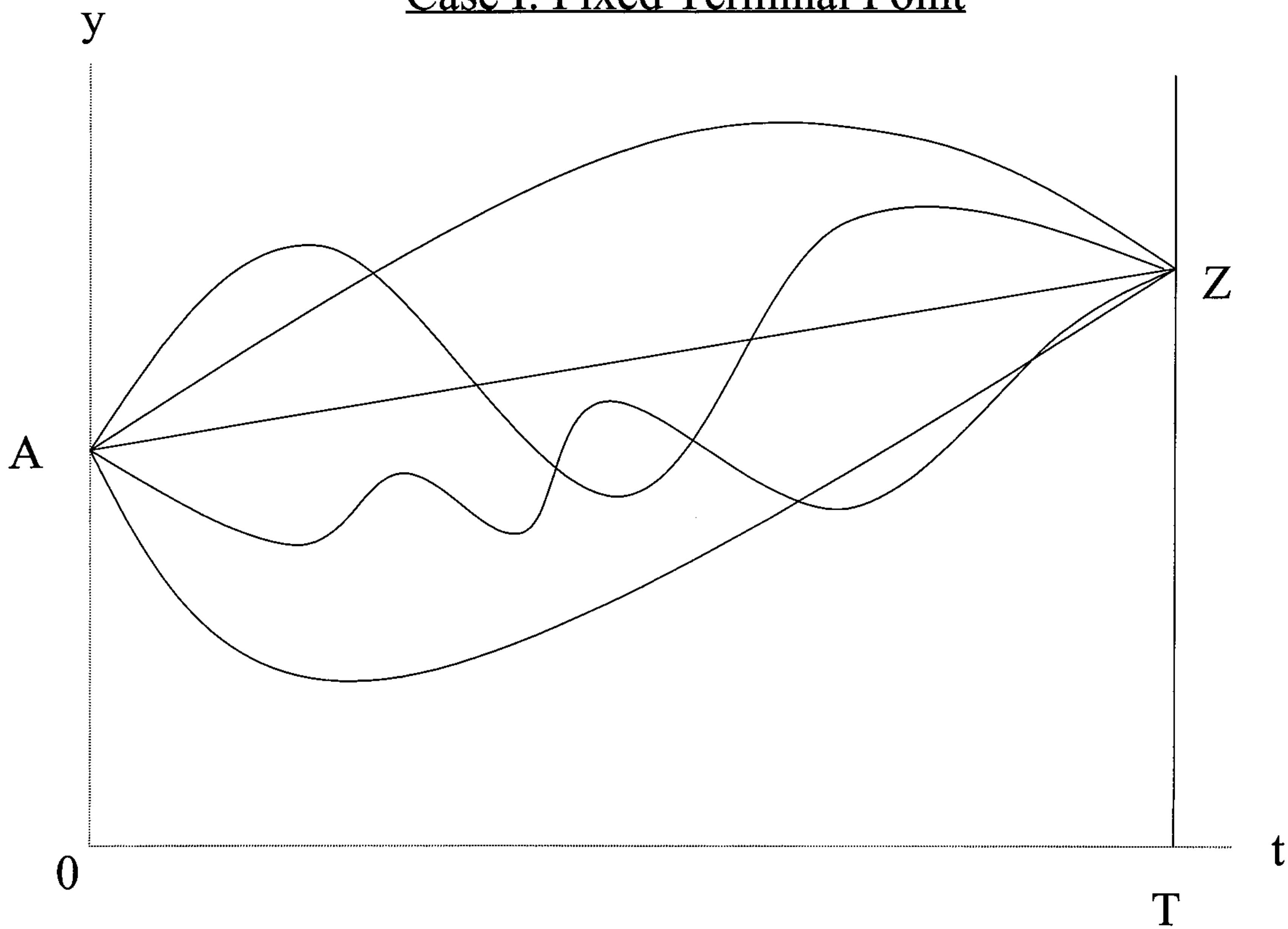
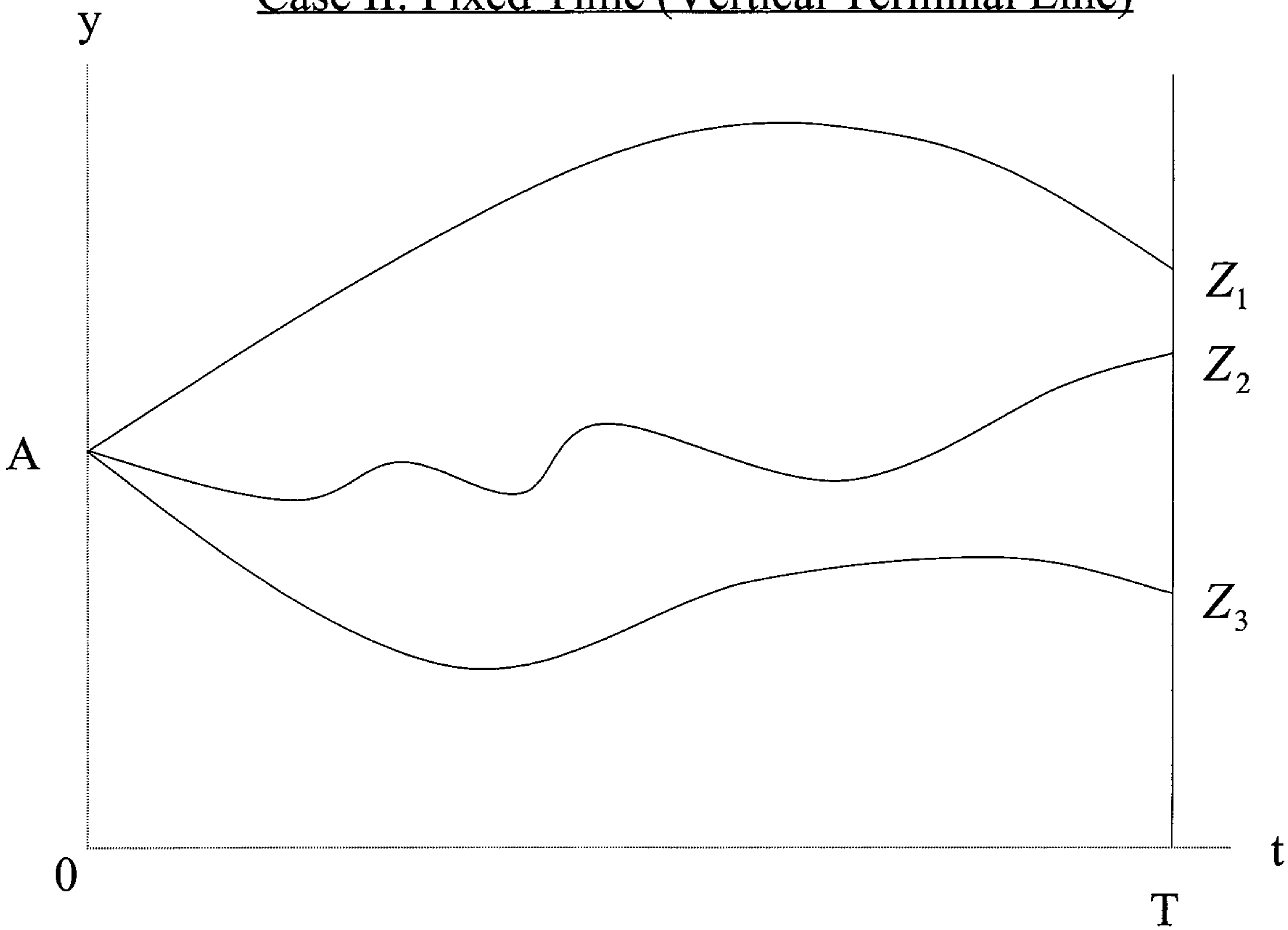
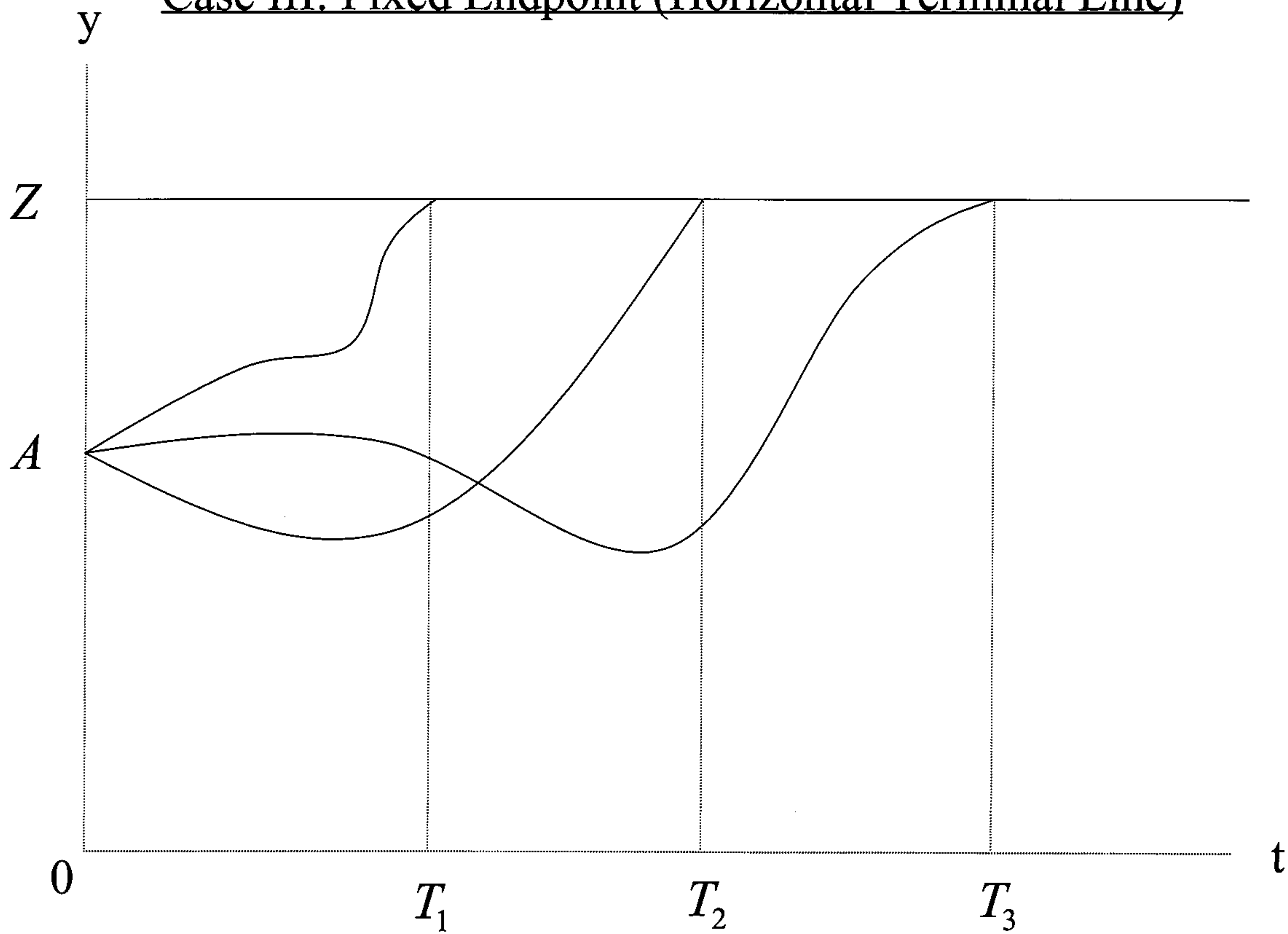
Case I: Fixed Terminal Point



Case II: Fixed Time (Vertical Terminal Line)



Case III: Fixed Endpoint (Horizontal Terminal Line)



Case IV: Terminal Curve)