Advanced Macroeconomics II Sample Exam

I. The basic mathematics.

1. Let $X = \mathbb{R}$ and let $\rho : X \to \mathbb{R}$ be defined as $\rho(x, y) = |x - y|$. Is the interval $(0, 1)$, regarded as a set in the metric space (X, ρ) closed?

2. Let $X = (0, 1)$. Let $\rho : X \to \mathbb{R}$ be defined as $\rho(x, y) = |x - y|$. Is the interval $(0, 1)$, regarded as a set in the metric space (X, ρ) closed? Is the metric space (X, ρ) complete?

3. Let the correspondence $G : [0, 1] \to \mathbb{R}$ be defined as $G(x) = x$ if $0 \leq x < 1$, and $G(x) = [0, 1]$ if $x = 1$. Is G u.h.c? Is G l.h.c?

4. Let the correspondence $G : [0, 1] \to \mathbb{R}$ be defined as $G(x) = [0, 1]$ if $0 \leq x < 1$, and $G(x) = 1$ if $x = 1$. Is G u.h.c? Is G l.h.c?

5. Let the function $f : [0, 1] \to \mathbb{R}$ be defined as $f(x) = 0.9x$. Is f a contraction mapping?

II. Consider an economy with many identical agents. (Think of that each point in the closed interval $[0, 1]$ corresponding to an agent.) There is one unit of tree. The tree generates d amount of fruit at the start of each period, and the fruit perishes at the end of the period. An agent's period utility function is U and his lifetime utility from consumption stream $(c_0, c_1, ...)$ is

$$\sum_{t=0}^{\infty} \beta^t U(c_{t+1}).$$

We assume that $0 < \beta < 1$ and that U is continuous, strictly increasing, strictly concave, bounded, continuously differentiable. The agent maximizes his lifetime utility. At period zero, the ownership is evenly distributed among agents. At period t, after fruit is mature, there is an open market for trading fruit and ownership of the tree. Consider an equilibrium in that the price of tree is p at each period.

1. Given price p, write down the agent's problem in the form of a Bellman equation.

2. Assume that there is a solution to the Bellman equation is continuous, strictly increasing, strictly concave, bounded, continuously differentiable. Write down the first order condition and the envelop condition associated to the Bellman equation.

3. Use the result from part 2 to determine the equilibrium price p.

III. The problem of optimal growth in a one-good economy can be written as

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U[f(k_t) - k_{t+1}], \tag{1}$$
$$\text{s.t. } 0 \leq k_{t+1} \leq f(k_t),\ t = 0, 1, ...,$$
$$\text{given } x_0 \geq 0.$$

We assume that $0 < \beta < 1$, that U is continuous, strictly increasing, strictly concave, bounded, continuously differentiable, and $U'(0) = \infty$, and that f is continuous, strictly increasing, concave, continuously differentiable, and $f(0) = 0$.

1. What is the Bellman equation corresponding to the problem (1)?

2. Show that there exists a unique continuous and bounded function $v : \mathbb{R}_+ \to \mathbb{R}$ that satisfies the above Bellman equation.

3. Show that v is strictly concave.

4. Show that v is strictly increasing.

5. Show that v is differentiable.

(In your argument, you can refer to results from theorems, corollaries, lemmas, and the relevant exercises in the textbook. For the current problem, an example of relevant exercise is Exercise 3.13 b, and Exercise 5 should not be regarded as relevant exercise. Of course, you should verify hypotheses involved by yourself.)

IV. Refer to the last problem (III). Let the correspondence $G : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as follows: If k is the amount of capital in the current period, then $G(k)$ is the set of optimal choices of capital carried to the next period.

1. Show that the correspondence G is u.h.c and compact valued.

2. Show that G is single valued.

3. Show that G is continuous.

According to parts 2 and 3, the correspondence G is actually a continuous function. So we write G as g to emphasize this fact.

4. Express the first order condition and the envelop condition associated to the Bellman equation (refer to III.1) in terms of v, U, f, β, and g.

5. Show that $0 < g(x_2) - g(x_1) < f(x_2) - f(x_1)$ if $x_2 > x_1$.